(1)

[This question paper contains 8 printed pages.]

Your Roll No. 2022

Sr. No. of Question Paper : 1150 **A**

Unique Paper Code : 32351401_LOCF

Name of the Paper : BMATH-408 Partial Differential Equation

Name of the Course : **CBCS B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **four** questions.
3. **All** questions carry equal marks.

SECTION – I

Attempt any **two** parts out of the following.

Marks of each part are indicated.

1. (a) Define the following with one example each : (6)

P.T.O.

(i) Quasi-linear first order partial differential equation (PDE).

(ii) Semi-linear first order PDE.

(iii) Linear first order PDE.

State whether the following first order PDE is quasi-linear, semi-linear, linear or non-linear :

$$\left(xy^2\right)u_x - \left(yx^2\right)u_y = u^2(x^2 - y^2)$$

Justify.

(b) Solve the Cauchy problem (6)

$$uu_x + u_y = 1$$

such that $u(s,0) = 0$, $x(s,0) = 2s^2$,

$$y(s,0) = 2s,\ s > 0.$$

(c) Obtain the solution of the pde (6)

$$x\left(y^2 + u\right)u_x - y\left(x^2 + u\right)u_y = (x^2 - y^2)u,$$

with the data u(x,y) = 1 on x + y = 0.

(d) Apply $\sqrt{u} = v$ and $v(x,y) = f(x) + g(y)$ to solve

$$x^4 u_x^2 + y^2 u_y^2 = 4u .$$

(6)

2. Attempt any **two** parts out of the following :

(a) Apply the method of separation of variables u(x,y) = f(x)g(y) to solve

$$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$$

such that $u(x,0) = 3e^{\frac{x^2}{4}}$. (6.5)

(b) Find the solution of the equation (6.5)

$$yu_x - 2xyu_y = 2xu$$

with the condition $u(0,y) = y^3$.

(c) Reduce into canonical form and solve for the general solution (6.5)

$$u_x - yu_y - u = 1.$$

(d) Derive the one-dimensional heat equation :

$$u_t = \kappa u_{xx},$$

where κ is a constant. (6.5)

SECTION – II

3. Attempt any **two** parts out of the following :

(a) Find the characteristics and reduce the equation $u_{xx} - (\sec h^4 x)u_{yy} = 0$ into canonical form. (6)

(b) Find the characteristics and reduce the equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xyu_x + y^2 u_y = 0$$

into canonical form. (6)

(c) Transform the equation $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$ to the form $v_{\xi\eta} = cv$, c=constant, by introducing the new variable $v = ue^{-(a\zeta + b\eta)}$, where a and b are undetermined coefficients. (6)

(d) Use the polar co-ordinates r and θ (x=r cosθ, y=r sinθ) to transform the Laplace equation $u_{xx} + u_{yy} = 0$ into polar form. (6)

4. Attempt any **two** parts out of the following :

(a) Find the D'Alembert solution of the Cauchy problem for one dimensional wave equation given by

$$u_{tt} - c^2 u_{xx} = 0, x \in R, t > 0$$
$$u(x,0) = f(x), x \in R,$$
$$u_t(x,0) = g(x), x \in R.$$

(6.5)

(b) Solve (6.5)

$$y^3 u_{xx} - y u_{yy} + u_y = 0,$$
$$u(x,y) = f(x) \text{ on } x + \frac{y^2}{2} = 4 \text{ for } 2 \le x \le 4,$$
$$u(x,y) = g(x) \text{ on } x - \frac{y^2}{2} = 0 \text{ for } 0 \le x \le 2,$$
$$\text{with } f(2) = g(2).$$

(c) Determine the solution of initial boundary value problem

$$u_{tt} = 16u_{xx}, \quad 0 < x < \infty, t > 0$$
$$u(x,0) = \sin x, \quad 0 \le x < \infty,$$
$$u_t(x,0) = x^2, \quad 0 \le x < \infty,$$
$$u(0,t) = 0, \quad t \ge 0.$$

(6.5)

(d) Determine the solution of initial boundary value problem (6.5)

$$u_{tt} = 9u_{xx}, \quad 0 < x, \infty, t > 0,$$
$$u(x,0) = 0, \quad 0 \le x < \infty,$$
$$u_t(x,0) = x^3, \quad 0 \le x < \infty$$
$$u_x(0,t) = 0, \quad t \ge 0.$$

SECTION – III

5. Attempt any **two** parts out of the following :

(a) Determine the solution of the initial boundary-value problem by method of separation of variables

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \ t > 0$$
$$u(x,0) = \begin{cases} hx/a, & 0 \le x \le a \\ h(l-x)/(l-a), & a \le x \le l \end{cases},$$
$$u_t(x,0) = 0, \quad 0 \le x \le l,$$
$$u(0,t) = 0 = u(l,t) = 0 \quad t \ge 0$$

(6.5)

(b) Obtain the solution of IBVP (6.5)

$$u_t = u_{xx}, \quad 0<x<2, \ t>0,$$
$$u(x,0)=x, \ 0 \le x \le 2,$$
$$u(0,t)=0, \quad u_x(2,t)=1, \quad t \ge 0,$$

(c) Determine the solution of the initial-value problem (6.5)

$$u_{tt} = c^2 u_{xx} + x^2,$$
$$u(x,0) = x, \quad 0 \le x \le 1,$$
$$u_t(x,0) = 0, \quad 0 \le x \le 1,$$
$$u(0,t) = 0, u(1,t) = 0, \quad t > 0.$$

(d) Determine the solution of the initial-value problem (6.5)

$$u_t = ku_{xx}, \quad 0 < x < 1, t > 0,$$
$$u(x,0) = x(1-x), \quad 0 \le x \le 1$$
$$u(0,t) = t, \quad u(1,t) = \sin t, \quad t > 0.$$

6. Attempt any **two** parts out of the following :

(a) Determine the solution of the initial boundary-value problem by method of separation of variables (6)

$$u_{tt} = c^2 u_{xx}, 0 < x < a, t > 0$$
$$u(x,0) = 0, \quad 0 \le x \le a,$$
$$u_t(x,0) = \begin{cases} \frac{v_0}{a} x, & 0 \le x \le a \\ v_0 (l-x)/(l-a), & a \le x \le l \end{cases}$$
$$u(0,t) = 0 = u(a,t) = 0, \quad t \ge 0.$$

(b) Find the temperature distribution in a rod of length l. The faces are insulated, and the initial temperature distribution is given by x (l - x). (6)

(c) Establish the validity of the formal solution of the initial boundary – value problem (6)

$$u_t = ku_{xx}, \quad 0 \le x \le l, \ t > 0,$$
$$u(x,0) = f(x), \qquad 0 \le x \le l,$$
$$u(0,t) = 0, \qquad t > 0,$$
$$u_x(1,t) = 0, \qquad t > 0.$$

(d) Prove the uniqueness of the solution of the problem : (6)

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \ t > 0,$$
$$u(x,0) = f(x), \ 0 \le x \le l,$$
$$u_t(x,0) = g(x), 0 \le x \le l,$$
$$u(0,t) = u(0,t) = 0, \ t > 0.$$

(2)

[This question paper contains 8 printed pages.]

Your Roll No. 2022

Sr. No. of Question Paper : 1377 **A**

Unique Paper Code : 32351402

Name of the Paper : BMATH-409; Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 hours + 30 minutes Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let f be a bounded function on [a, b]. Define integrability of f on [a, b] in the sense of Riemann. (6)

(b) Prove that every continuous function on [a, b] is integrable. Discuss about the integrability of discontinuous functions. (6)

(c) Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on [–1, 1], Show that $\left|\int_{-1}^{1} f(t)dt\right| \leq 2$. (6)

(d) Let f(x) = x for rational x; and f(x) = 0 for irrational x. Calculate the upper and lower Darboux integrals of f on the interval [0, b]. Is f integrable on [0, b]? (6)

2. (a) State Fundamental Theorem of Calculus II. Use it to calculate

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt. \quad (6.5)$$

(b) Let f be defined as (6.5)

$$f(t)=\begin{cases}0, & t<0\\ t, & 0\leq t\leq 1\\ 4, & t>1\end{cases}$$

(i) Determine the function $F(x)=\int_0^x f(t)dt$.

(ii) Sketch F. Where is F continuous?

(iii) Where is F differentiable? Calculate F′ at points of differentiability.

(c) State and prove Intermediate value theorem for Integral Calculus. Give an example to show that condition of continuity of the function cannot be relaxed. (6.5)

(d) Let f be a continuous function on $\mathbb{R}$. Define

$$G(x)=\int_0^{\sin x} f(t)dt \text{ for } x \in \mathbb{R}.$$

Show that G is differentiable on $\mathbb{R}$ and compute G′. (6.5)

3. (a) Let $\beta(p, q)$ (where $p, q > 0$) denotes the beta function, show that

$$\beta(p,q)=\int_{0^+}^{\infty}\frac{u^{p-1}}{(1+u)^{p+q}}du=\int_{0^+}^{1}\frac{v^{p-1}+v^{q-1}}{(1+v)^{p+q}}dv. \quad (6)$$

(b) Determine the convergence and divergence of the following improper, integrals

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ (6)

(c) Define Improper Integral of type II.

Show that the improper integral $\int_1^{\infty} \frac{dx}{x^p}$ converges iff $p > 1$. (6)

(d) Show that the improper integral $\int_{\pi}^{\infty} \frac{\text{Sin } x}{x} dx$ is convergent but doesn't converge absolutely. (6)

4. (a) Let $< f_n >$ be a sequence of integrable functions on $[a,b]$ and suppose that $< f_n >$ converges uniformly on [a, b] to f. Show that is f is integrable. (6)

(b) Define

(i) pointwise convergence of sequence of functions

(ii) uniform convergence of a sequence of functions

(iii) If $A \subseteq \mathbb{R}$ and $\varnothing: A \to \mathbb{R}$ then define uniform norm of $\varnothing$ on A. (6)

(c) (i) Discuss the pointwise and uniform convergence of $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}, n \in N$.

(ii) Show that the sequence $< f_n >$ where $f_n(x) = \frac{n}{x+n}$, $x \geq 0$ is uniformly convergent in any finite interval. (6)

(d) (i) Show that the sequence $< f_n >$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ uniformly convergent on $[0, \pi]$.

(ii) Discuss the pointwise and uniform convergence of the sequence $g_n(x) = x^n$ for $x \in \mathbb{R}$, $n \in N$. (6)

5. (a) (i) State and prove the Cauchy Criteria for uniform convergence of series. (3.5)

(ii) Show that the series $\sum_{0}^{\infty}(1-x)x^n$ is not uniformly convergent on [0, 1] (3)

(b) Show that $\sum \frac{(-1)^n}{n^p} \frac{x^{2n}}{(1+x^{2n})}$ converges absolutely and uniformly for all values of x if $p > 1$.

(c) Is the sequence $< f_n >$ where $f_n = \frac{\sin(nx+n)}{n}$, uniformly convergent on $\mathbb{R}$? Justify. (6.5)

(d) If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in N$ and $\sum f_n$ converges to f uniformly on D then prove that f is continuous on D. (6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the following power series :

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}(x+1)^n$

(ii) $\sum_{n=0}^{\infty} x^{n!}$ (6.5)

(b) Let f(x) = $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Show that the function f is differentiable on (–R, R) and

$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$. (6.5)

(c) Show that

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - - - -$

for $|x| < 1$

(ii) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - - -$. (6.5)

(d) Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, show that the power series converges uniformly on $[-R_1, R_1]$. Also, show that the sum function $f(x)$ is continuous on the interval $(-R, R)$. (6.5)

(3)

[This question paper contains 6 printed pages.]

Your Roll No. 2022

Sr. No. of Question Paper : 1395 **A**

Unique Paper Code : 32351403

Name of the Paper : BMATH-410 – Ring Theory and Linear Algebra – I

Name of the Course : **CBCS (LOCF) B.Sc. (H) Mathematics**

Semester : IV

Duration : 3.30 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Define zero divisors in a ring. Let R be the set of all real valued functions defined for all real numbers under function addition and multiplication. Determine all zero divisors of R. (6½)

(b) What is nilpotent element? If a and b are nilpotent elements of a commutative ring, show that a + b is also nilpotent. Give an example to show that this may fail if the ring R is not commutative. (6½)

(c) Let R be a commutative ring with unity. Prove that U(R), the set of all units of R, form a group under multiplication of R. (6½)

(d) Determine all subrings of $\mathbb{Z}$, the set of integers. (6½)

2. (a) Define centre of a ring. Prove that centre of a ring R is a subring of R. (6)

(b) Suppose R is a ring with $a^2 = a$, for all $a \in R$. Show that R is a commutative ring. (6)

(c) Show that any finite field has order p^n, where p is prime. (6)

(d) Let R be a ring with unity 1. Prove that if 1 has infinite order under addition, then CharR = 0, and if 1 has order n under addition, then CharR = n. (6)

3. (a) Let R be a commutative ring with unity and let A be an ideal of R. Then show that R/A is a field if and only if A is maximal ideal. (6½)

(b) Prove that $I = <2 + 2i>$ is not prime ideal of $\mathbb{Z}[i]$, How many elements are in $\frac{\mathbb{Z}[i]}{I}$? What is the characteristic of $\frac{\mathbb{Z}[i]}{I}$? (6½)

(c) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) = 0\}$. Prove that I is not a maximal ideal. (6½)

(d) Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients and let $< x^2 + 1 >$ denote the principal ideal generated by $x^2 + 1$. Then show that

$$\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} = \{g(x) + \langle x^2+1\rangle \mid g(x) \in \mathbb{R}[x] =$$

$$\{ax + b + \langle x^2+1\rangle \mid a, b \in \mathbb{R}\}$$

(6½)

4. (a) If R is a ring with unity and the characteristic of R is $n > 0$, then show that R contains a subring isomorphic to $\mathbb{Z}_n$ and if the characteristic of R is 0 then R contains a subring isomorphic to $\mathbb{Z}$. (6)

(b) Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. (6)

(c) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 11 if and only if $a_0 - a_1 + a_2 - \cdots + (-1)^k a_k$ is divisible by 11. (6)

(d) Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism. (6)

5. (a) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is a smallest subspace of V that contains both W_1 and W_2. (6)

(b) For the following polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as linear combination of other two. (6)

$$\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}.$$

(c) Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k < n)$. (6)

(d) Let $W_1 = \{(a, b, 0) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$ and $W_2 = \{(0, b, c)$ E $\mathbb{R}^3 : b, c \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Determine $\dim(W_1)$, $\dim(W_2)$, $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$. Hence deduce that $W_1 + W_2 = \mathbb{R}^3$. Is $\mathbb{R}^3 = W_1 \oplus W_2$? (6)

6. (a) Let V and W be finite-dimensional vector spaces having ordered bases β and γ respectively, and let $T : V \rightarrow W$ be linear. Then for each $u \in V$, show

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta .$$ (6½)

(b) Let $T: P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ be linear transformation defined by

$$T(a + bx + cx^2) = (a - c) + (a - c)x + (b - a)x^2 + (c - b)x^3.$$

Find null space N(T) and range space R(T). Also verify Rank-Nullity Theorem. (6½)

(c) For the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$ and ordered basis $\beta = \{(1, 1, 0), (0, 1, 1), (1, 2, 2)\}$, find $[L_A]_\beta$. Also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$. (6½)

(d) Let V and W be vector spaces and let $T: V \to W$ be linear. Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. (6½)